Extremal Hypergraphs and Combinatorial Geometry

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ABSTRACT. Here we overview some of the methods and results of extremal graph and hypergraph theory. A few geometric applications are also given.

1. Introduction and notation

Most combinatorial problems can be formulated as (extremal) hypergraph problems. Extremal hypergraph theory applies a broad array of tools and results from other fields like number theory, linear and commutative algebra, probability theory, geometry, and information theory. On the other hand, it has a number of interesting applications in all parts of combinatorics, and in geometry, integer programming, and computer science. Some recent successes include: the best upper bound for the number of unit distances in a convex polygon [40]; the first nontrivial upper bound for the number of halving hyperplanes [3]; and the counterexample to the longstanding Borsuk's conjecture by Kahn and Kalai [48].

We overview some of the methods used in extremal graph and hypergraph theory and illustrate them by Turán-type problems. Some geometric applications are also given; more can be found in the recent monograph [60].

A *hypergraph* H is a pair $H = (V, \mathcal{E})$, where V is a finite set, the set of *vertices*, and $\mathcal{E}$ is a family of subsets of V, the set of *edges*. If all the edges have r elements, then H is called an *r-graph*, or r-uniform hypergraph. The complete r-partite hypergraph $\mathcal{K}_{t_1,t_2,\dots,t_r}$ has a partition of its vertex set $V = V_1 \cup \ldots \cup V_r$, such that $|V_i| = t_i$, and $\mathcal{E} = \{E : |E \cap V_i| = 1\}$ for all $1 \le i \le r$. The set $\{1, 2, \ldots, n\}$ is abbreviated as $[n]$.

2. The Turán problem

Given a graph F, what is $\mathrm{ex}(n, F)$, the maximum number of edges of a graph with n vertices not containing F as a subgraph? This problem was proposed for $F = C_4$ by Erdős [19] in 1938 and in general by Turán [72]. For example, $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ (Mantel [57], Turán [72]). The Erdős-Stone-Simonovits [29], [26] theorem says that the order of magnitude of $\mathrm{ex}(n, F)$ depends only on the chromatic number, $\lim_{n\to\infty} \mathrm{ex}(n, F)/\binom{n}{2} = 1 - (\chi(F) - 1)^{-1}$. This gives a sharp estimate, except for bipartite graphs.

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Very little is known even about simple cases when F is a fixed even cycle C_{2k} or a fixed complete bipartite graph $\mathcal{K}_{k,k}$. For a survey of extremal graph problems, see Bollobás' book [5], or Simonovits [67], [66]. For Turán problems for hypergraphs see [41].

3. Minimum graphs of given girth

Erdős proved in 1959 that for any $\chi \geq 2$ and $g \geq 3$ there exists a graph of chromatic number χ and girth g. (The *girth* is the length of the shortest cycle.) Known elementary constructions yield graphs with an enormous number of vertices. Recently, deep results in number theory combined with the eigenvalue methods in graph theory have been invoked with success to explicitly construct relatively small graphs, called *Ramanujan graphs*, with large chromatic number and girth (Margulis [58], Imrich [47], and Lubotzky, Phillips, and Sarnak [56]). These graphs give the lower bound in the following inequality:

$$\Omega(n^{1+(3/(4k+21))}) \leq \mathrm{ex}(n, C_{2k}) \leq 90kn^{(k+1)/k}. \tag{1}$$

The first nontrivial lower bound, $\Omega(n^{1+(1/2k)})$, was given by Erdős (see in [28]) using probabilistic methods. The upper bound is due to Bondy and Simonovits [6] and is believed to give the correct order of magnitude.

Constructions giving $\Omega(n^{1+(1/k)})$ are known only for $k = 2, 3$, and 5 (Benson [4]). Wenger [74] simplified these cases. Recently Lazebnik, Ustimenko, and Woldar gave new algebraic constructions [53] for all k.

4. Bipartite graphs

For every bipartite graph F that is not a forest there is a positive constant c (not depending on n) such that $\Omega(n^{1+c}) \leq \mathrm{ex}(n, F) \leq O(n^{2-c})$. The lower bound follows from (1). The upper bound is provided by the following result of Kővári, Sós, and Turán [50] concerning the complete bipartite graph.

$$\mathrm{ex}(n, \mathcal{K}_{t,t}) < \frac{1}{2}(t-1)^{1/t}n^{2-(1/t)} + (t-1)n/2 = O(n^{2-(1/t)}). \tag{2}$$

This bound gives the right order of magnitude of $\mathrm{ex}(n, \mathcal{K}_{t,t})$ for $t = 2$ and $t = 3$ and probably for all t. For $t > 3$ the best lower bound, $\mathrm{ex}(n, \mathcal{K}_{t,t}) \geq \Omega(n^{2-2/(t+1)})$, is due to Erdős and Spencer [28]. Until now the only asymptotic for a bipartite graph that is not a forest, $\mathrm{ex}(n, C_4) = \frac{1}{2}(1 + o(1))n^{3/2}$, was due to Erdős, Rényi, and T. Sós [25] and to Brown [8]. This has recently been generalized [43]:

THEOREM 1 *For any fixed* $t \geq 1$ $\quad \mathrm{ex}(n, \mathcal{K}_{2,t+1}) = \frac{1}{2}\sqrt{t}n^{3/2} + O(n^{4/3})$.

A large graph with no $\mathcal{K}_{2,t+1}$. The following algebraic construction is closely related to the examples for C_4-free graphs and is inspired by an example of Hyltén-Cavallius [46] and Mörs [59] given for Zarankiewicz's problem [76]. Let q be a prime power such that $(q-1)/t$ is an integer. We construct a $\mathcal{K}_{2,t+1}$-free graph G on $(q^2-1)/t$ vertices such that every vertex has degree q or $q-1$. Let $\mathbf{F}$ be the

q-element field, $h \in \mathbf{F}$ an element of order t, $H = \{1, h, h^2, \ldots, h^{t-1}\}$. The vertices of G are the t-element orbits of $(\mathbf{F} \times \mathbf{F}) \setminus (0,0)$ under the action of multiplication by powers of h. Two classes $\langle a, b\rangle$ and $\langle x, y\rangle$ are joined by an edge in G if $ax + by \in H$.

Note that the sets $N\langle a, b\rangle = \{\langle x, y\rangle : ax + by \in H\}$ form a q-uniform, symmetric, solvable, group divisible t-design.
Brown [8] gave an algebraic construction to show $\mathrm{ex}(n, \mathcal{K}_{3,3}) \geq (1/2 - o(1))n^{5/3}$. Very recently, it was shown to be asymptotically optimal [44].

THEOREM 2 $\quad \mathrm{ex}(n, \mathcal{K}_{t,t}) \leq \frac{1}{2}(1 + o(1))n^{2-(1/t)}$.

5. The number of unit distances

What is the maximum number of times, $f^{(d)}(n)$, that the same distance can occur among pairs of n points in the d-dimensional space $\mathbf{R}^d$? The complete bipartite graph $\mathcal{K}_{2,3}$ cannot be realized on the plane, so $f^{(2)}(n) \leq \mathrm{ex}(n, \mathcal{K}_{2,3}) = O(n^{3/2})$. Erdős [20] conjectured in 1945 that the grid gave the best value, $f^{(2)}(n) = O(n^{1+C/\log\log n})$. Spencer, Szemerédi, and Trotter [69] proved $f^{(2)}(n) \leq O(n^{4/3})$. A new proof appeared in Clarkson et al. [13]. Erdős observed that for the 3-space $n^{4/3}\log\log n \leq f^{(3)}(n) \leq \mathrm{ex}(n, \mathcal{K}_{3,3}) = O(n^{5/3})$. The best upper bound is due to Clarkson et al. [13], $f^{(3)}(n) \leq O(n^{3/2}\beta(n))$, where $\beta(n)$ is an extremely slowly growing function related to the inverse of Ackermann's function.

It is proved in [40], using the Turán theory of matrices resembling the Davenport-Schinzel problem solved by Sharir [64], that the maximum number of unit distances in a *convex* n-gon, $g^{(2)}(n)$, is at most $7n\log n$. Erdős and Moser [24] conjecture that $g^{(2)}(n)$ is linear. Edelsbrunner and Hajnal [17] showed that $g^{(2)}(n) \geq 2n - 4$.

6. The number of halving planes

In most problems an estimate on the number of sub(hyper)graphs isomorphic to a given structure F is more applicable than the information about the Turán number $\mathrm{ex}(n, F)$. Rademacher proved in 1941 that a graph with $\lfloor n^2/4\rfloor + 1$ edges has at least $\lfloor n/2\rfloor$ triangles (see Lovász and Simonovits [55]). The best, in most cases almost optimal, lower bound for the number of triangles in a graph of n vertices and e edges was given by Fisher [31]. The following theorem was proved in [21] in an implicit form. For more explicit formulations see Erdős and Simonovits [27] or Frankl and Rödl [37].

THEOREM 3 (Erdős [21]) *For any positive integers r and $t_1 \leq \cdots \leq t_r$ there exist positive constants c' and c'' such that the following holds. If an r-graph has n vertices and $e \geq c'n^{r-\alpha}$ edges, where $\alpha = 1/(t_1 t_2 \cdots t_{r-1})$, then it contains at least*

$$c''(e/n^r)^{t_1 t_2 \cdots t_r} n^{t_1 + \cdots + t_r}$$

copies of the complete r-hypergraph $\mathcal{K}_{t_1,\ldots,t_r}$.

Let $S \subset \mathbf{R}^3$ be an n-set in general position. A plane containing three of the points is called a *halving plane* if it dissects S into two parts of (almost) equal cardinality. In [3] it was proved that the number of halving planes is at most $O(n^{2.998})$. As a main tool, for every set Y of n points in the plane, a set N of size $O(n^4)$ is constructed such that the points of N are distributed almost evenly in the triangles determined by Y. The proof is a combined application of Turán theory (Theorem 3), the random method, and fractional hypergraph coverings.

A generalization of Tverberg's theorem [73], conjectured in [3], was proved by Živaljević and Vrećica [77] by Lovász' topological method [54]. The exponent 2.998 was improved most recently by Dey and Edelsbrunner [14] to 8/3. The best lower bound is $\Omega(n^2 \log n)$. The best 2-dimensional upper bound is due to Pach, Szemerédi, and Steiger [61].

7. Intersecting hypergraphs

Here we consider the more general hypergraph problems, where the forbidden configurations are k-uniform hypergraphs. For example, if the excluded hypergraph consists of two disjoint edges; i.e., the family $\mathcal{H}$ of k-sets is intersecting, then $|\mathcal{H}| \leq \binom{n-1}{k-1}$ for $n \geq 2k$, where n stands for the number of vertices. If $\mathcal{G}$ is a family of k-sets of $[n]$ such that any two members intersect in at least t elements, then $|\mathcal{G}| \leq \binom{n-t}{k-t}$, provided n is sufficiently large, $n > n_0(k,t)$. Equality holds if and only if $\mathcal{G}$ consists of all k-element subsets of $[n]$ containing a fixed t-element subset (Erdős, Ko, and Rado [23]). The exact value of $n_0(k,t) = (k-t+1)(t+1)$ was determined by Frankl [32] (for $t \geq 15$), and by Wilson [75] (for all t, using association schemes). Define

$$\mathcal{A}^r = \{G \in \binom{[n]}{k} : |G \cap [t+2r]| \geq t+r\}.$$

$|\mathcal{A}^r|$ is the largest among the $\mathcal{A}^i$'s if $(k-t+1)(2+\frac{t-1}{r+1}) \leq n < (k-t+1)(2+\frac{t-1}{r})$.

CONJECTURE 1 (Erdős, Ko, and Rado [23]; Frankl [32]) *If $\mathcal{G}$ is a t-intersecting family of maximum cardinality, then $\mathcal{G}$ is isomorphic to $\mathcal{A}^r$ for some r.*

This conjecture was proved [35] for $r < c\sqrt{t \log t}$, where $c > 0.02$ is an absolute constant. The proof is a triumph of the transformation method (left shifting).

THEOREM 4 [34] *Suppose that a k-uniform hypergraph on n vertices has more than $\binom{n-t-1}{k-t-1}$ edges, $k \geq 2t+2$, $n > n_1(k)$. Then it contains two edges F, F' such that $|F \cap F'| = t$.*

8. Prescribed intersections

Let $0 \leq \ell_1 < \ell_2 < \cdots < \ell_s < k \leq n$ be integers. The family $\mathcal{G} \subseteq \binom{V}{k}$ is an $(n,k,\{\ell_1,\ldots,\ell_s\})$-system if $|G \cap G'| \in \{\ell_1,\ldots,\ell_s\}$ holds for every $G, G' \in \mathcal{G}$, $G \neq G'$. Denote $\{\ell_1,\ldots,\ell_s\}$ by L and let us denote by $m(n,k,L)$ the maximum cardinality of an (n,k,L)-system. The determination of $m(n,k,L)$ is the simplest looking Turán-type problem; the family of forbidden configurations consists only of hypergraphs of size two. The most well-known result of this type is the Erdős-Ko-Rado theorem dealing with the case $L = \{t, t+1, \ldots, k-1\}$ (see above).

The problem of determining $m(n, k, L)$ for general L was proposed by Larman [51], and first studied by Deza, Erdős, and Frankl [15]. A few years earlier yet, Ray-Chaudhuri and Wilson [10] proved a very general upper bound, namely that $m(n, k, L) \le \binom{n}{s}$ holds for all $n \ge k$ and $|L| = s$. The proof uses linear algebraic independence of some higher order incidence matrices over the reals. This was generalized for finite fields by Frankl and Wilson [39]. Very recently Frankl, Ota, and Tokushige have determined almost all the 8192 exponents of the $m(n, k, L)$'s up to $k \le 12$. The complexity of these questions can be seen in the following result of Frankl [33]. For every rational $r \ge 1$ there exist k and L such that $m(n, k, L) = \Theta(n^r)$. The proof of this combines the Δ-system method, and algebraic and geometric constructions. A similar conjecture of Erdős and Simonovits [27] for graphs is still open: for every rational $1 < p/q < 2$ there exists a bipartite graph G with $\mathrm{ex}(n, G) = \Theta(n^{p/q})$, and every bipartite graph has a rational exponent r with $\mathrm{ex}(n, G) = \Theta(n^r)$.

Improving a result of Babai and Frankl [1] a necessary and sufficient condition for $m(n, k, L) = \Theta(n)$ has been found. We say that the numbers $\ell_1, \ldots, \ell_s$ and k satisfy property $(*)$ if there exists a family $\mathcal{I} \subset 2^{[k]}$, closed under intersection, such that $\cup \mathcal{I} = [k]$ and $|I| \in L$ for all $I \in \mathcal{I}$.

THEOREM 5 [41] *If $(*)$ is satisfied, then $m(n, k, L) > (1/8k)n^{k/(k-1)}$. On the other hand, if $(*)$ does not hold, then $m(n, k, L) \le (2^{k^k})n$.*

9. The chromatic number of the space

The following problem was proposed by Hadwiger [45]. What is the minimum number $c(n)$ such that $\mathbf{R}^n$ can be divided into $c(n)$ subsets $\mathbf{R}^n = C_1 \cup \ldots \cup C_{c(n)}$ such that no pair of points within the same C_i is at unit distance? In other words, what is the chromatic number of the unit distance graph? This problem is wide open even in the plane, we have only $4 \le c(2) \le 7$. The regular simplex shows $c(n) \ge n+1$; the first nonlinear lower bound $\Omega(n^2)$ was given by Larman and Rogers [52]. They also gave an exponential upper bound of 3^n. The above-mentioned forbidden intersection theorems of Frankl and Wilson [39] easily lead to a lower bound of 1.2^n.

Sixty years ago Borsuk [7] raised the following question. Is it true that every set of diameter one in $\mathbf{R}^d$ can be partitioned into $d+1$ sets of diameter smaller than one? The following theorem of Frankl and Rödl [38] led to the counterexample given by Kahn and Kalai [48]. Let n be an integer divisible by four, and let $\mathcal{F}$ be a family of subsets of an n-element underlying set such that no two sets in the family have intersection of size $n/4$. Then $|\mathcal{F}| < 1.99^n$.

10. Szemerédi's regularity lemma

This is a powerful graph-approximation method. We need some notation. Let G be an arbitrary, fixed graph. For two disjoint subsets $V_1, V_2 \subset V(G)$, let $E(V_1, V_2)$ denote the set of edges of G with one endpoint in V_1 and the other in V_2. The *edge-density* between these sets is

$$\delta(V_1, V_2) = \frac{|E(V_1, V_2)|}{|V_1| \cdot |V_2|}.$$

The pair (V_1, V_2) is called *ε-regular*, if $|\delta(V_1', V_2') - \delta(V_1, V_2)| < \varepsilon$ holds for all $V_1' \subset V_1$ and $V_2' \subset V_2$ whenever $|V_1'| \geq \varepsilon |V_1|$ and $|V_2'| > \varepsilon |V_2|$.

THEOREM 6 (Szemerédi's regularity lemma [70]) *For every $0 < \varepsilon < 1$ and for every integer r there exists an $M(\varepsilon, r)$ such that the following is true for* every *graph G. The vertex set of G can be partitioned into ℓ classes $V_1, \ldots, V_\ell$ for some $r \leq \ell \leq M(\varepsilon, r)$ so that these classes are almost equal (i.e., $||V_i| - |V(G)|/\ell| < 1$), and all but at most $\varepsilon \ell^2$ pairs (V_i, V_j) are ε-regular.*

The main feature of Theorem 6 is that it allows us to handle any given graph as if it were a random one. Even the most chaotic graph can be decomposed into a relatively small number of almost regular systems. Rödl [62] and Elekes [18] showed that one cannot require all pairs (V_i, V_j) to be ε-regular. Sós and Simonovits [68] (joining to works of Thomason [71] and Chung, Graham, and Wilson [12]) used Theorem 6 to describe the so-called *quasi-random* sequences of graphs. This connection is illuminated in the next section.

11. Graphs with a small number of triangles

This is an application of Szemerédi's regularity lemma. Let F be a fixed graph on the k-element vertex set $\{u_1, \ldots, u_k\}$, and suppose that the graph G on n vertices contains only $o(n^k)$ copies of F. We will prove that one can delete $o(n^2)$ edges from G to eliminate all copies of F. Reformulating this statement without o's for the special case $F = K_3$ we get

THEOREM 7 *For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the following holds: For every graph G on n vertices with at most δn^3 triangles, one can find a set E' with at most εn^2 edges, such that $G \setminus E'$ is triangle-free.*

The theorem says that it is impossible to distribute *evenly* a small number of triangles in a graph with a large number of edges.

Let $V_1, \ldots, V_k$ be disjoint m-element sets and let $0 \leq \delta \leq 1$. A random graph on the vertex set $V_1 \cup \ldots \cup V_k$ is defined by choosing every pair of vertices $u \in V_i$, $v \in V_j$ with probability δ. The expected edge-density between V_i and V_j is δ. Moreover, the expected number of copies of F such that $v_i \in V_i$, and $v_i v_j$ is connected for $u_i u_j \in E(F)$ is $\delta^{|E(F)|} m^k$. The next lemma is used (sometimes in implicit form) in most applications of the Regularity Lemma.

LEMMA 1 *Let $a_1, \ldots, a_k$ be natural numbers, $\sum a_i = p$, let F be a graph on the p-element vertex set $\{u_{ij} : 1 \leq i \leq k, 1 \leq j \leq a_i\}$, and let $0 < \varepsilon < p^{-p}$, $\varepsilon^{1/p} \leq \delta < 1/2$. Suppose that the graph G has k pairwise disjoint subsets $V_1, \ldots, V_k \subset V(G)$, $|V_i| \geq m_i$ for all $1 \leq i \leq k$, and $\delta(V_i', V_j') \geq \delta$ hold for all $V_i' \subset V_i$, $V_j' \subset V_j$ if for some $1 \leq i < j \leq k$ there is an edge $u_{ia} u_{jb} \in E(F)$ and $|V_i'| \geq \varepsilon |V_i|$, $|V_j'| \geq \varepsilon |V_j|$. Then the subgraph of G induced by $V_1 \cup \ldots \cup V_k$ contains at least $\delta^{|E(F)|} 2^{-p} \prod (m_i)^{a_i}$*

copies (embeddings) of F with vertex sets $\{v_{ia}\}$ such that $v_{ia} \in V_i$, $v_{jb} \in V_j$, and $v_{ia}v_{jb} \in E(G)$ for $u_{ia}u_{jb} \in E(F)$.

Proof. The case $a_1 = \cdots = a_k = 1$ implies the general case. Indeed, choose a_i disjoint (m_i/a_i)-element sets from V_i, and apply the lemma for this new partition ($\varepsilon^* = \varepsilon/(\max a_i)$, $\delta^* = \delta$). Then do this for all possible partitions. Finally, the case $p = k$ follows by induction on k, as was done for $F = K_k$ in [60]. □

Proof of Theorem 7. Let $\varepsilon_0 = (\varepsilon/3)^k$. Suppose that $\varepsilon_0 < k^{-k}/3$, and define $r = \lceil 3/\varepsilon_0 \rceil$. We claim that $\delta = (2M(r,\varepsilon_0))^{-k}(\varepsilon_0)^{|E(F)|/k}$ will suffice. Let G be an arbitrary graph with at most δn^k copies of F. Apply Szemerédi's lemma with the above r and ε_0. We get a partition $V_1, \ldots, V_\ell$. Delete all edges covered by any V_i, then delete all edges connecting V_i and V_j if the pair (V_i, V_j) is not ε_0-regular, or if its density is less than $\varepsilon_0^{1/k}$. We have deleted at most $n^2/\ell + \varepsilon_0 n^2 + \varepsilon_0^{1/k} n^2$ edges. Then the rest of the graph is F-free; otherwise, the lemma would provide us at least $\varepsilon_0^{|E(F)|/k}(n/2\ell)^k$ copies. □

12. The maximum number of edges in a minimal graph of diameter 2

A graph G of diameter 2 is minimal if the deletion of any edge increases its diameter. Murty and Simon (see in [9]) conjecture that such a G cannot have more than $n^2/4$ edges. This was proved for $n > n_0$ in [42] in the following slightly stronger form: the only extremum is the complete bipartite graph. The value of n_0 is explicitly computable, but the proof gives a vastly huge number, a tower of 2's of height about 10^{14}.

This theorem is the first application of Szemerédi's regularity lemma yielding an exact answer (at least for $n > n_0$). Bounds were given by Caccetta and Häggkvist [9] and Fan [30]. It is easy to see that the theorem gives a direct generalization of Turán's triangle theorem. (If every edge in G that is contained in a triangle is also contained in some minimal path of length 2, then $|E(G)| \leq n^2/4$.)

The proof utilizes the following Turán type result of Ruzsa and Szemerédi [63]: if $\mathcal{F}$ is a triangle-free, 3-uniform hypergraph on n vertices (that means that no 6 vertices carry more than 2 triples), then $|\mathcal{F}| = o(n^2)$. In almost all other applications of Theorem 6 one only needs the Ruzsa-Szemerédi theorem. Note that it is an easy corollary of Theorem 7. (Replacing each triple by the 3 pairs contained in it one gets a graph with $3|\mathcal{F}|$ edges and only $|\mathcal{F}| \leq \binom{n}{2} = o(n^3)$ triangles.) Other short proofs and generalizations for r-uniform hypergraphs (also based on Theorem 6) were given by Erdős, Frankl, and Rödl [22] and by Duke and Rödl [16].

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